Chapter 2: Differentiation

Section 2.1: The Derivative and the Tangent Line Problem

In algebra, you learned how to calculate the slope ($\Delta y/\Delta x$) of a line between two points on a curve, but the question of how to find the slope of a line tangent to a curve at a *single* point (see **Figure 1**) was left unanswered.

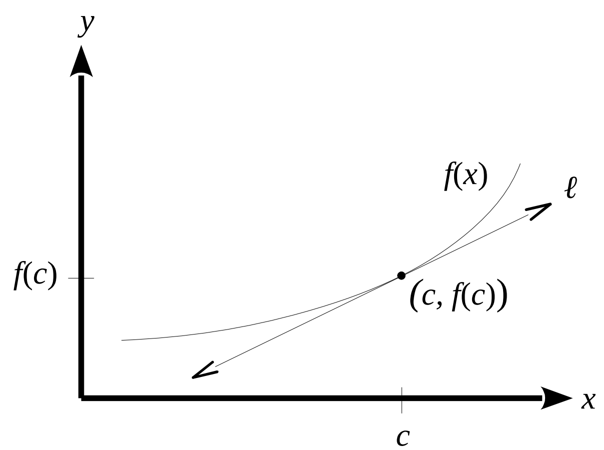

Figure 1: Line ℓ is tangent to $f(x)$ at the point $(c, f(c))$

In calculus, one may approach the latter question by calculating the slope between two points (c, $f(c)$) and (x, $f(x)$) (**Figure 2**), and taking the limit as they approach each other (**Figure 3**)

$$\lim_{x \to c} \frac{\Delta y}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

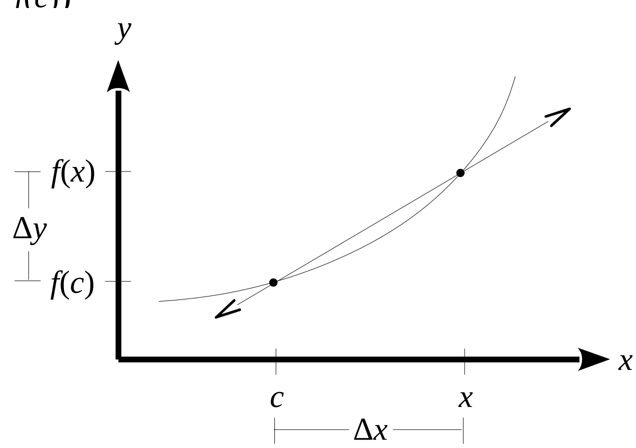

Figure 2: Slope of the secant line connecting two points

It is often more convenient to rewrite this expression in terms of the difference between the points (Δx). Hence,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This expression has a special name in calculus and is known as the *derivative* of $f(x)$ (with respect to x) and is symbolized *df*/*dx.*

$$\boxed{\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

You may also use one of these alternative notations for the derivative of $f(x)$:

$$\frac{df}{dx} = \frac{d}{dx} f(x) = f'(x) = D_x f(x)$$

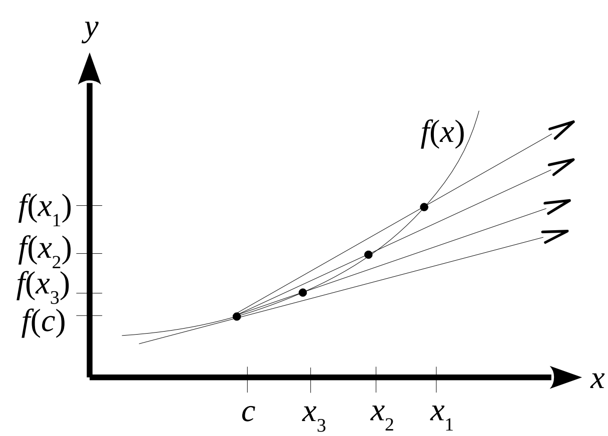

Figure 3: The limit of the slope between two points as x approaches c equals the slope of the tangent line at $x = c$.

<u>Differentiability and Continuity</u>

If the derivative (according to the above definition) exists at a particular point x, the function is said to be *differentiable* at that point.

For a function to be differentiable over an entire interval, it is required that it be also continuous over that interval. The reason is that derivatives involve limits, and limits do not exist at discontinuities.

All differentiable functions are continuous. The converse of this statement is not true, however: *not all continuous functions are differentiable.* There is even one function—the *Weierstrass function*—that is continuous everywhere but differentiable nowhere!

All polynomial functions, as well as sin *x* and cos *x,* are differentiable everywhere.

Power functions whose exponents are less than 1, such as $f(x) = x^{1/3}$, are not differentiable when $x = 0$, because the slope approaches infinity near the origin.

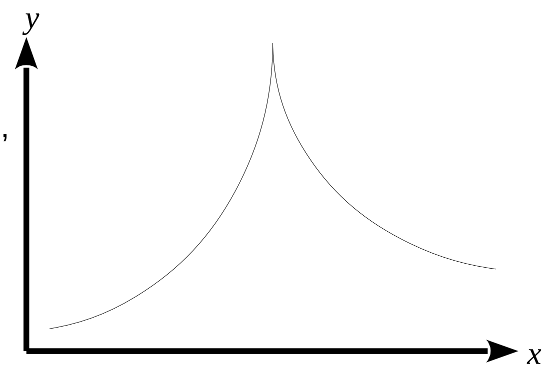

Figure 4: An example of a function which is continuous but not differentiable at a point.

Section 2.2: Basic Differentiation Rules and Rates of Change

This section contains introduces several important shortcuts that can be used to calculate derivatives without resorting to the definition. All of these can be derived from the original definition of the derivative. What follows is the definitions of each rule, followed by a short paraphrase that you can use to remember it:

The *constant rule* states that if *c* is a constant, then

$$\boxed{\frac{d}{dx}c = 0}$$

"The derivative of a constant is zero."

The *power rule* states that

$$\boxed{\frac{d}{dx}x^n = n\,x^{n-1}}$$

"Bring the exponent to the front and reduce by one."

The *constant multiple rule* states that if *c* is a constant, then

$$\boxed{\frac{d}{dx}[c\,f(x)] = c\frac{d}{dx}f(x)}$$

"Constants may be factored out of the derivative."

The *sum and differences rule* state that

$$\boxed{\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)}$$

"The derivative of a sum is equal to the sum of the derivatives."

The derivatives of sin *x* and cos *x* are

$$\boxed{\frac{d}{dx}\sin x = \cos x} \quad \boxed{\frac{d}{dx}\cos x = -\sin x}$$

"The derivative of sin x is cos x. The derivative of cos x is -sin x."

Rates of change

A more general interpretation of the derivative is the rate of change of one variable with respect to another. In other words, *df/dx* means "how fast *f* is changing when *x* is varied." In particular, the derivative *ds/dt*—where *s* is displacement and *t* is time—represents velocity. (As you may have learned in physics, speed is the absolute value of velocity.)

The exact equations governing the relationship between displacement and velocity depend on the displacement function. In the particular case of constant acceleration, the equation is:

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0$$

where v_0 and s_0 are the velocity and displacement when t = 0 and a is acceleration. In the case of an free-falling object, $a = g$ -9.8 m/s^2.

Section 2.3: Product and Quotient Rules and Higher-Order Derivatives

The Product Rule

The *product rule* states that

$$\boxed{\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)}$$

"The derivative of a product is the derivative of the first times the second plus the derivative of the second times the first."

This is sometimes abbreviated as $(fg)' = f'g + g'f$

The *quotient rule* states that

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}}$$

"The derivative of a quotient is the derivative of the numerator time the numerator minus the derivative of the denominator times the numerator all over the denominator squared."

In other words, $(f/g)' = \frac{f'g - g'f}{g^2}$

Because many complicated functions can be written as the product or quotient of two simpler functions, the product and quotient rules vastly increase the number of functions that can be differentiated.

Higher-order derivatives

To calculate the second *derivative* of a function, first calculate its derivative and then take the derivative again. The second derivative can be written in the following ways:

$$y'' = f''(x) = \frac{d^2 y}{dx^2} = \frac{d^2}{dx^2} f(x) = D_x^{\ 2}(y)$$

This process of repeated derivatives can be repeated an indefinite number of times. In general, the *n*th derivative is expressed as follows:

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} f(x) = D_x^{\ n}\, y$$

One familiar second derivative is *acceleration*, which is the first derivative of velocity with respect to time, and the second derivative of the displacement with respect to time.

$$\boxed{a = \frac{dv}{dx} = \frac{d^2 s}{dt^2}}$$